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On the bottom of spectra under coverings

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Abstract For a Riemannian covering $M_1 \to M_0$ of connected Riemannian manifolds with respective fundamental groups $\Gamma_1 \subseteq \Gamma_0$, we show that the bottoms of the spectra of M_0 and M_1 coincide if the right action of Γ_0 on $\Gamma_1 \backslash \Gamma_0$ is amenable.

Keywords Bottom of spectrum · Amenable covering

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1 Introduction

In this article, we study the behaviour under coverings of the bottom of the spectrum of Schrödinger operators on Riemannian manifolds.

Let M be a connected Riemannian manifold, not necessarily complete, and $V : M \to \mathbb{R}$ be a smooth potential with associated *Schrödinger operator* $\Delta + V$. We consider $\Delta + V$ as an unbounded symmetric operator in the space $L^2(M)$ of square integrable functions on M with domain $C_c^\infty(M)$, the space of smooth functions on M with compact support.

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For a non-vanishing Lipschitz continuous function on M with compact support in M, we call

$$R(f) = \frac{\int_M (|\nabla f|^2 + Vf^2)}{\int_M f^2} \tag{1.1}$$

the *Rayleigh quotient of* f. We let

$$\lambda_0(M, V) = \inf R(f), \tag{1.2}$$

where f runs through all non-vanishing Lipschitz continuous functions on M with compact support in M. If $\lambda_0(M, V) > -\infty$, then $\Delta + V$ is bounded from below on $C_c^\infty(M)$ and $\lambda_0(M, V)$ is equal to the bottom of the spectrum of the *Friedrichs extension of* $\Delta + V$. If $\lambda_0(M, V) = -\infty$, then the spectrum of any self-adjoint extension of $\Delta + V$ is not bounded from below.

Recall that $\Delta + V$ is essentially self-adjoint on $C_c^\infty(M)$ if M is complete and $\inf V > -\infty$. Then the unique self-adjoint extension of $\Delta + V$ is its closure. In the case where M is the interior of a complete Riemannian manifold N with smooth boundary and where V extends smoothly to the boundary of N, $\lambda_0(M, V)$ is equal to the bottom of the Dirichlet spectrum of $\Delta + V$ on N.

In the case of the *Laplacian*, that is, $V = 0$, we also write $\lambda_0(M)$ and call it the *bottom of the spectrum of* M. It is well known that $\lambda_0(M)$ is the supremum over all $\lambda \in \mathbb{R}$ such that there is a positive smooth λ-eigenfunction $f: M \to \mathbb{R}$ (see, e.g., [3, Theorem 7], [4, Theorem 1], or [5, Theorem 2.1]). It is crucial that these eigenfunctions are not required to be square-integrable. In fact, $\lambda_0(M)$ is exactly the border between the positive and the L^2 spectrum of Δ (see, e.g., [5, Theorem 2.2]).

Suppose now that M is simply connected and let $\pi_0: M \to M_0$ and $\pi_1: M \to M_1$ be Riemannian subcovers of M. Let Γ_0 and Γ_1 be the groups of covering transformations of π_0 and π_1, respectively, and assume that $\Gamma_1 \subseteq \Gamma_0$. Then the resulting Riemannian covering $\pi: M_1 \to M_0$ satisfies $\pi \circ \pi_1 = \pi_0$. Let $V_0: M_0 \to \mathbb{R}$ be a smooth potential and set $V_1 = V_0 \circ \pi$.

Since the lift of a positive λ-eigenfunction of Δ on M_0 to M_1 is a positive λ-eigenfunction of Δ, we always have $\lambda_0(M_0) \le \lambda_0(M_1)$ by the above characterization of the bottom of the spectrum of Δ by positive eigenfunctions. In Sect. 4, we present a short and elementary proof of the inequality which does not rely on the characterization of λ_0 by positive eigenfunctions:

Theorem 1.1 *For any Riemannian covering* $\pi: M_1 \to M_0$ *as above,*

$$\lambda_0(M_0, V_0) \le \lambda_0(M_1, V_1).$$

Brooks showed in [2, Theorem 1] that $\lambda_0(M_0) = \lambda_0(M_1)$ in the case where M_0 is complete, has *finite topological type*, and π is *normal* with *amenable* group $\Gamma_1 \backslash \Gamma_0$ of covering transformations. Bérard and Castillon extended this in [1, Theorem 1.1] to $\lambda_0(M_0, V_0) = \lambda_0(M_1, V_1)$ in the case where M_0 is complete, $\pi_1(M_0)$ is finitely generated [this assumption occurs in point (1) of their Section 3.1], and the right action of Γ_0 on $\Gamma_1 \backslash \Gamma_0$ is amenable. We generalize these results as follows:

Theorem 1.2 *If the right action of* Γ_0 *on* $\Gamma_1 \backslash \Gamma_0$ *is amenable, then*

$$\lambda_0(M_0, V_0) = \lambda_0(M_1, V_1).$$

Here a right action of a countable group Γ on a countable set X is said to be *amenable* if there exists a Γ-invariant mean on $L^\infty(X)$. This holds if and only if the action satisfies the

Følner condition: For any finite subset $G \subseteq \Gamma$ and $\varepsilon > 0$, there exists a non-empty, finite subset $F \subseteq X$, a *Følner set*, such that

$$|F \backslash Fg| \leq \varepsilon |F| \tag{1.3}$$

for all $g \in G$. By definition, Γ is *amenable* if the right action of Γ on itself is amenable, and then any action of Γ is amenable.

In comparison with the results of Brooks, Bérard, and Castillon, the main point of Theorem 1.2 is that we do not need any assumptions on metric and topology of M_0. A main new point of our arguments is that we adopt our constructions more carefully to the different competitors for λ_0 separately.

2 Fundamental domains and partitions of unity

Choose a complete Riemannian metric h on M_0. In what follows, geodesics, distances, and metric balls in M_0, M_1, and M are taken with respect to h and its lifts to M_1 and M, respectively.

Fix a point x in M_0. For any $y \in \pi^{-1}(x)$, let

$$D_y = \{z \in M_1 \mid d(z, y) \leq d(z, y') \text{ for all } y' \in \pi^{-1}(x)\} \tag{2.1}$$

be the *fundamental domain* of π centered at y. Then D_y is closed in M_1, the boundary ∂D_y of D_y has measure zero in M_1, and $\pi \colon D_y \backslash \partial D_y \to M_0 \backslash C$ is an isometry, where C is a subset of the cut locus $\mathrm{Cut}(x)$ of x in M_0. Recall that $\mathrm{Cut}(x)$ is of measure zero. Moreover, $M_1 = \cup_{y \in \pi^{-1}(x)} D_y$, $y \in \pi^{-1}(x)$.

Lemma 2.1 *For any $\rho > 0$, there is an integer $N(\rho)$ such that any z in M_1 is contained in at most $N(\rho)$ metric balls $B(y, \rho)$, $y \in \pi^{-1}(x)$.*

Proof Let $z \in B(y_1, \rho) \cap B(y_2, \rho)$ with $y_1 \neq y_2$ in $\pi^{-1}(x)$ and $\gamma_1, \gamma_2 \colon [0, 1] \to M_1$ be minimal geodesics from y_1 to z and y_2 to z, respectively. Then $\sigma_1 = \pi \circ \gamma_1$ and $\sigma_2 = \pi \circ \gamma_2$ are geodesic segments from x to $\pi(z)$. Since $y_1 \neq y_2$, σ_1 and σ_2 are not homotopic relative to $\{0, 1\}$. Hence, if z lies in in the intersection of n pairwise different balls $B(y_i, \rho)$ with $y_1, \dots, y_n \in \pi^{-1}(x)$, then the concatenations $\sigma_1^{-1} * \sigma_i$ represent n pairwise different homotopy classes of loops at x of length at most 2ρ. Hence n is at most equal to the number $N(\rho)$ of homotopy classes of loops at x with representatives of length at most 2ρ. □

Lemma 2.2 *If $K \subseteq M_0$ is compact, then $\pi^{-1}(K) \cap D_y$ is compact. More precisely, if $K \subseteq B(x, r)$, then $\pi^{-1}(K) \cap D_y \subseteq B(y, r)$.*

Proof Choose $r > 0$ such that $K \subseteq B(x, r)$. Let $z \in \pi^{-1}(K) \cap D_y$ and γ_0 be a minimal geodesic from $\pi(z) \in K$ to x. Let γ be the lift of γ_0 to M_1 starting in z. Then γ is a minimal geodesic from z to some point $y' \in \pi^{-1}(x)$. Since $z \in D_y$, this implies

$$d(z, y) \leq d(z, y') \leq L(\gamma) = L(\gamma_0) < r.$$

Hence $\pi^{-1}(K) \cap D_y \subseteq B(y, r)$. □

Let $K \subseteq M_0$ be a compact subset and choose $r > 0$ such that $K \subseteq B(x, r)$. Let $\psi \colon \mathbb{R} \to \mathbb{R}$ be the function which is equal to 1 on $(-\infty, r]$, to $t + 1 - r$ for $r \leq t \leq r + 1$, and to 0 on $[r + 1, \infty]$. For $y \in \pi^{-1}(x)$, let $\psi_y = \psi_y(z) = \psi(d(z, y))$. Note that $\psi_y = 1$ on $\pi^{-1}(K) \cap D_y$ and that $\operatorname{supp} \psi_y = \bar{B}(y, r + 1)$.

Lemma 2.3 *Any z in M_1 is contained in the support of at most $N(r+1)$ of the functions ψ_y, $y \in \pi^{-1}(x)$.*

Proof This is clear from Lemma 2.1 since supp ψ_y is contained in the ball $B(y, r+1)$. □

In particular, each point of M_1 lies in the support of only finitely many of the functions ψ_y. Therefore the function $\psi_1 = \max\{1 - \sum \psi_y, 0\}$ is well defined. By Lemma 2.2, we have $\operatorname{supp} \psi_1 \cap \pi^{-1}(K) = \emptyset$. Together with ψ_1, the functions ψ_y lead to a partition of unity on M_1 with functions φ_1 and φ_y, $y \in \pi^{-1}(x)$, given by

$$\varphi_1 = \frac{\psi_1}{\psi_1 + \sum_{z \in \pi^{-1}(x)} \psi_z} \quad \text{and} \quad \varphi_y = \frac{\psi_y}{\psi_1 + \sum_{z \in \pi^{-1}(x)} \psi_z}. \tag{2.2}$$

Note that $\operatorname{supp} \varphi_1 = \operatorname{supp} \psi_1$ and $\operatorname{supp} \varphi_y = \operatorname{supp} \psi_y$ for all $y \in \pi^{-1}(x)$.

Lemma 2.4 *The functions φ_y, $y \in \pi^{-1}(x)$, are Lipschitz continuous with Lipschitz constant $3N(r+1)$.*

Proof The functions ψ_y, $y \in \pi^{-1}(x)$, are Lipschitz continuous with Lipschitz constant 1 and take values in $[0, 1]$. Hence ψ_1 is Lipschitz continuous with Lipschitz constant $N = N(r+1)$, by Lemma 2.3, and takes values in $[0, 1]$. Therefore the denominator $\chi = \psi_1 + \sum_{z \in \pi^{-1}(x)} \psi_z$ in the fraction defining the φ_y is Lipschitz continuous and takes values in $[1, N]$. Hence

$$\begin{aligned}
|\varphi_y(z_1) - \varphi_y(z_2)| &\le \frac{|(\chi(z_2) - \chi(z_1))\psi_y(z_1) + \chi(z_1)(\psi_y(z_1) - \psi_y(z_2))|}{\chi(z_1)\chi(z_2)} \\
&\le \frac{(2N + N)d(z_1, z_2)}{\chi(z_1)\chi(z_2)} \le 3Nd(z_1, z_2).
\end{aligned}$$

□

As a consequence of Lemma 2.4, we get that $\varphi_1 = 1 - \sum \varphi_y$ is also Lipschitz continuous with Lipschitz constant $6N(r+1)^2$.

3 Pulling up

Let f be a non-vanishing Lipschitz continuous function on M_0 with compact support and let $f_1 = f \circ \pi$. We will construct a cutoff function χ on M_1 such that $R(\chi f_1)$ is close to $R(f)$.

Let g be the given Riemannian metric on M_0 and h be a complete background Riemannian metric on M_0 as in Sect. 2. Then there is a constant $A \ge 1$ such that

$$A^{-1} g \le h \le A g \tag{3.1}$$

on the support of f. We continue to take distances and metric balls in M_0, M_1, and M with respect to h and its respective lifts to M_1 and M.

Fix a point x in M_0. With $K = \operatorname{supp} f$ and $r > 0$ such that $K \subseteq B(x, r)$, we get a partition of unity with functions φ_1 and φ_y, $y \in \pi^{-1}(x)$, as above.

Fix preimages $u \in M$ and $y = \pi_1(u) \in M_1$ of x under π_0 and π, respectively. Write $\pi_0^{-1}(x) = \Gamma_0 u$ as the union of Γ_1-orbits $\Gamma_1 g u$, where g runs through a set R of representatives of the right cosets of Γ_1 in Γ_0, that is, of the elements of $\Gamma_1 \backslash \Gamma_0$. Then $\pi^{-1}(x) = \{\pi_1(gu) \mid g \in R\}$. Let

$$\begin{aligned} S &= \{s \in R \mid d(y, \pi_1(su)) \le 2r + 2\} \\ &= \{s \in R \mid d(u, tsu) \le 2r + 2 \text{ for some } t \in \Gamma_1\}, \\ T &= \{t \in \Gamma_1 \mid d(u, tsu) \le 2r + 2 \text{ for some } s \in S\}, \\ G &= TS \subseteq \Gamma_0. \end{aligned}$$

Since the fibres of π and π_0 are discrete, S and T are finite subsets of Γ_0, hence also G.

Let $\varepsilon > 0$ and $F \subseteq \Gamma_1 \backslash \Gamma_0$ be a Følner set for G and ε satisfying (1.3). Let

$$P = \{g \in R \mid \Gamma_1 g \in F\} \subseteq R$$

and set

$$\chi = \sum_{g \in P} \varphi_{\pi_1(gu)}.$$

Since $|P| = |F| < \infty$, supp χ is compact. Hence, by Lemma 2.4, χf_1 is compactly supported and Lipschitz continuous on M_1. Let

$$Q = \{y \in \pi^{-1}(x) \mid (\chi f_1)(z) \neq 0 \text{ for some } z \in D_y\}.$$

To estimate the Rayleigh quotient of χf_1, it suffices to consider χf_1 on the union of the D_y, $y \in Q$. We first observe that

$$P_1 = \{\pi_1(gu) \mid g \in P\} \subseteq Q.$$

To show this, let $y = \pi_1(gu)$ and observe that f_1 does not vanish identically on $\pi^{-1}(K) \cap D_y$ and that φ_y is positive on $\pi^{-1}(K) \cap D_y$. Since R is a set of representatives of the right cosets of Γ_1 in Γ_0, there exists a one-to-one correspondence between P and P_1, and hence

$$|P| = |P_1| \le |Q|.$$

The problematic subset of Q is

$$Q_- = \{y \in Q \mid 0 < \chi(z) < 1 \text{ for some } z \in \pi^{-1}(K) \cap D_y\}.$$

Let now $y \in Q_-$ and $z \in \pi^{-1}(K) \cap D_y$ with $0 < \chi(z) < 1$. Since $\pi_1(gu)$, $g \in R$, runs through all points of $\pi^{-1}(x)$, we have $\sum_{g \in R} \varphi_{\pi_1(gu)}(z) = 1$. Hence there are $g_1, \ldots, g_k \in R \backslash P$ such that $\varphi_{\pi_1(g_i u)}(z) \neq 0$ and

$$\chi(z) + \sum \varphi_{\pi_1(g_i u)}(z) = 1.$$

Furthermore, there has to be a $g \in P$ with $\varphi_{\pi_1(gu)}(z) \neq 0$. Then the supports of the functions $\varphi_{\pi_1(gu)}$ and $\varphi_{\pi_1(g_i u)}$ intersect and we get $d(\pi_1(gu), \pi_1(g_i u)) \le 2r + 2$. That is, we have $d(gu, h_i g_i u) \le 2r + 2$ for some $h_i \in \Gamma_1$. We conclude that

$$d(u, g^{-1} h_i g_i u) = d(gu, h_i g_i u) \le 2r + 2.$$

Since π_1 is distance non-increasing, we get that there are $s_i \in S$ and $t_i \in T$ such that $g^{-1} h_i g_i = t_i s_i$, and then $h_i g_i = g t_i s_i$. Since $g_i \notin P$, we conclude that $\Gamma_1 g t_i s_i \notin F$, i.e., $\Gamma_1 g \in F \backslash F(t_i s_i)^{-1}$. Since $(t_i s_i)^{-1} \in G$, there are at most $\varepsilon |F||G|$ such elements $g \in P$. Since $d(y, z) \le r$ and $d(z, \pi_1(gu)) \le r + 1$, we conclude with Lemma 2.1 that for fixed $g \in P$ there are at most $N(2r + 1)$ such $y \in Q$. We conclude that

$$\begin{aligned} |Q_-| &\le \varepsilon |F||G| N(2r + 1) \\ &= \varepsilon |P||G| N(2r + 1) \le \varepsilon |Q||G| N(2r + 1). \end{aligned} \tag{3.2}$$

We now estimate the Rayleigh quotient of χf_1. For any $y \in Q_+ = Q \backslash Q_-$, we have $\chi = 1$ on $\pi^{-1}(K) \cap D_y$ and therefore

$$\begin{aligned}\int_{D_y} \{|\nabla(\chi f_1)|^2 + V_1(\chi f_1)^2\} &= \int_{D_y} \{|\nabla f_1|^2 + V_1 f_1^2\} \\ &= \int_{M_0} \{|\nabla f|^2 + V_0 f^2\}\end{aligned}$$

and

$$\int_{D_y} \chi^2 f_1^2 = \int_{D_y} f_1^2 = \int_{M_0} f^2,$$

where, here and below, integrals, gradients, and norms are taken with respect to the original Riemannian metric g on M.

For any $y \in Q_-$, we have

$$\int_{D_y} \chi^2 f_1^2 \le \int_{M_0} f^2 \quad \text{and} \quad \int_{D_y} |V_1|\chi^2 f_1^2 \le C_0 \int_{M_0} f^2,$$

where C_0 is the maximum of $|V_0|$ on supp $f = K$. By Lemma 2.3, Lemma 2.4, and (3.1), we have $|\nabla\chi|^2 \le 9N(r+1)^4 A$ on the support of f. Therefore

$$\begin{aligned}\int_{D_y} |\nabla(\chi f_1)|^2 &\le 2\int_{D_y} \{|\nabla\chi|^2 f^2 + \chi^2 |\nabla f \circ \pi|^2|\} \\ &\le 18N(r+1)^4 A \int_{M_0} f^2 + 2\int_{M_0} |\nabla f|^2.\end{aligned}$$

In conclusion,

$$\int_{D_y} \{|\nabla(\chi f_1)|^2 + |V_1|\chi^2 f_1^2\} \le C$$

for any $y \in Q_-$, where $C > 0$ is an appropriate constant, which depends on f, but not on y or the choice of ε and F. With $D = |G|N(2r+1)$, we obtain from (3.2) that

$$|Q_-| \le \frac{\varepsilon D}{1 - \varepsilon D} |Q_+|,$$

and conclude that

$$\begin{aligned}R(\chi f_1) &= \frac{\int \{|\nabla(\chi f_1)|^2 + V_1 \chi^2 f_1^2\}}{\int (\chi f_1)^2} \\ &= \frac{\sum_{y \in Q} \int_{D_y} \{|\nabla f_1|^2 + V_1 f_1^2\}}{\sum_{y \in Q} \int_{D_y} f_1^2} \\ &\le \frac{\sum_{y \in Q_+} \int_{D_y} \{|\nabla f_1|^2 + V_1 f_1^2\} + \varepsilon C D |Q_+|/(1 - \varepsilon D)}{\sum_{y \in Q_+} \int_{D_y} f_1^2} \\ &= \frac{\int_{M_0} \{|\nabla f|^2 + V_0 f^2\} + \varepsilon C D/(1 - \varepsilon D)}{\int_{M_0} f^2} \\ &= R(f) + \frac{\varepsilon C D}{(1 - \varepsilon D) \int_{M_0} f^2}.\end{aligned}$$

For $\varepsilon \to 0$, the right hand side converges to $R(f)$.

Proof of Theorem 1.2 By Theorem 1.1, we have $\lambda_0(M_0, V_0) \leq \lambda_0(M_1, V_1)$. By (1.2), the bottom of the spectrum of Schrödinger operators is given by the infimum of corresponding Rayleigh quotients $R(f)$ of Lipschitz continuous functions with compact support. The arguments above show that, for any such function f on M_0 and any $\delta > 0$, there is a Lipschitz continuous function χf_1 on M_1 with compact support and Rayleigh quotient at most $R(f) + \delta$. Therefore we also have $\lambda_0(M_0, V_0) \geq \lambda_0(M_1, V_1)$. □

4 Pushing down

Let f be a Lipschitz continuous function on M_1 with compact support. Define the *push down* $f_0 \colon M_0 \to \mathbb{R}$ of f by

$$f_0(x) = \left(\sum_{y \in \pi^{-1}(x)} f(y)^2 \right)^{1/2}.$$

Since supp f is compact, the sum on the right hand side is finite for all $x \in M_0$, and hence f_0 is well defined. We have supp $f_0 = \pi(\text{supp } f)$, and hence supp f_0 is compact. Furthermore, f_0 is differentiable at each point x, where f is differentiable at all $y \in \pi^{-1}(x)$ and $f(y) \neq 0$ for some $y \in \pi^{-1}(x)$, and then

$$\nabla f_0(x) = \frac{1}{f_0(x)} \sum_{y \in \pi^{-1}(x)} f(y) \pi_*(\nabla f(y)).$$

For the norm of the differential of f_0 at x, we get

$$\begin{aligned} |\nabla f_0(x)|^2 &\leq \frac{1}{f_0(x)^2} \left| \sum_{y \in \pi^{-1}(x)} f(y) \pi_*(\nabla f(y)) \right|^2 \\ &\leq \frac{1}{f_0(x)^2} \sum_{y \in \pi^{-1}(x)} f(y)^2 \sum_{y \in \pi^{-1}(x)} |\nabla f(y)|^2 \\ &= \sum_{y \in \pi^{-1}(x)} |\nabla f(y)|^2. \end{aligned}$$

Furthermore, f_0 is differentiable with vanishing differential at almost any point of $\{f_0 = 0\}$. Therefore f_0 is Lipschitz continuous and

$$\int_{M_0} f_0^2 = \int_{M_1} f^2, \quad \int_{M_0} V_0 f_0^2 = \int_{M_1} V_1 f^2, \quad \int_{M_0} |\nabla f_0|^2 \leq \int_{M_1} |\nabla f|^2.$$

In particular, we have $R(f_0) \leq R(f)$.

Proof of Theorem 1.1 For any non-vanishing Lipschitz continuous function f on M_1 with compact support, the push down f_0 as above is a Lipschitz continuous function on M_0 with compact support and Rayleigh quotient $R(f_0) \leq R(f)$. The asserted inequality follows now from the characterization of the bottom of the spectrum by Rayleigh quotients as in (1.2). □

5 Final remarks

It is well-known that any countable group is the fundamental group of a smooth four-manifold. (A variant of the usual argument for finitely presented groups, taking connected sums of $S^1 \times S^3$ and performing surgeries, can be used to produce five-manifolds with fundamental group any countable group.) In particular, for a non-finitely generated, amenable group G, e.g., $G = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ or $G = \mathbb{Q}$, there is a smooth manifold M with $\pi_1(M) \cong G$. In contrast to the results in [1,2], our main result also applies to such examples.

Moreover, we do not assume $\lambda_0(M_0, V_0) > -\infty$. Given any non-compact manifold M_0, it is indeed easy to construct a smooth potential V_0 such that $\lambda_0(M_0, V_0) = -\infty$. In fact, it suffices that $V_0(x)$ tends to $-\infty$ sufficiently fast as $x \to \infty$.

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